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Darboux transformation and soliton solutions for the Boiti–Pempinelli–Tu (BPT) hierarchy

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Abstract

Starting from a spectral problem, we derive the well-known Boiti–Pempinelli–Tu (BPT) hierarchy. An explicit and universal Darboux transformation for the whole hierarchy is constructed. The soliton solutions for the BPT hierarchy are obtained by applying the Darboux transformation.

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1. Introduction

The investigation of the exact solutions of nonlinear evolution equations plays an important role in the study of nonlinear physical phenomena. For example, the wave phenomena observed in fluid dynamics, plasma and elastic media are often modelled by the bell-shaped sech solutions and the kink-shaped tanh solutions. The exact solutions, if available, of those nonlinear equations facilitates the verification of numerical solvers and aids in the stability analysis of solutions. In the past decades, there has been a significant progress in the development of various methods. Among them, the Darboux transformation is a powerful method to get exact solutions of nonlinear partial differential equations. The key for constructing Darboux transformation is to expose kinds of covariant properties that the corresponding spectral problems possess. There have been many tricks to do this for getting explicit solutions to various soliton equations, including the KdV equation, KP equation, Davey–Stewartson equation, Yang-Mills flows, etc [1–8].

In this paper, we are interested in the Darboux transformation and exact solutions of the Boiti–Pempinelli–Tu (BPT) hierarchy associated with the following BPT spectral problem:

$$\psi_x = U\psi = \frac{1}{2}\begin{pmatrix} \lambda + \dfrac{s}{\lambda} & q + \dfrac{r}{\lambda} \\ q - \dfrac{r}{\lambda} & -\lambda - \dfrac{s}{\lambda} \end{pmatrix}\psi. \tag{1.1}$$

This spectral problem was first introduced by Boiti, Pempinelli and Tu in [9], where they considered a particular case of the more general spectral problem proposed by Boiti and Tu in [10] and they showed that the soliton equations of the related hierarchy are Hamiltonian systems with commuting flows on a symplectic Kahler manifold.

The outline of our present paper is as follows. In section 2, we derive the BPT hierarchy associated with the spectral problem (1.1). In section 3, we construct the Darboux transformation for the BPT hierarchy. In section 4, we construct soliton solutions for the BPT hierarchy by using its Darboux transformation.

2. The BPT hierarchy

In order to derive the isospectral hierarchy associated with (1.1), we consider the auxiliary problem

$$\psi_{t_n} = V^{(n)}\psi = \sum_{j=0}^{n}(A_j\lambda^{2(n-j)+1} + B_j\lambda^{2(n-j)})\psi, \tag{2.1}$$

where

$$A_j = \frac{1}{2}\begin{pmatrix} a_j & c_j \\ -c_j & -a_j \end{pmatrix}, \qquad B_j = \frac{1}{2}\begin{pmatrix} 0 & b_j \\ b_j & 0 \end{pmatrix}.$$

The compatibility condition between (1.1) and (2.1) yields the zero-curvature equation

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0, \tag{2.2}$$

which is equivalent to the following recurrence relations

$$\begin{pmatrix} 1 & 0 & 0 \\ q & 0 & \partial \\ \partial & -1 & q \end{pmatrix}\begin{pmatrix} c_{j+1} \\ b_{j+1} \\ a_{j+1} \end{pmatrix} = \begin{pmatrix} -s & \partial & r \\ 0 & r & 0 \\ 0 & s & 0 \end{pmatrix}\begin{pmatrix} c_j \\ b_j \\ a_j \end{pmatrix}, \qquad 0 \leqslant j \leqslant n-1 \tag{2.3}$$

and the equations

$$\begin{pmatrix} q_{t_n} \\ r_{t_n} \\ s_{t_n} \end{pmatrix} = \begin{pmatrix} c_{n+1} \\ -sb_n \\ -rb_n \end{pmatrix}. \tag{2.4}$$

Further we choose $a_0 = 1, b_0 = q, c_0 = 0$. From (2.3) we can easily prove that $c_j|_{(q,r)=(0,0)} = b_j|_{(q,r)=(0,0)} = 0 (1 \leqslant j \leqslant n)$. We also need to use the condition $a_j|_{(q,r)=(0,0)} = 0 (1 \leqslant j \leqslant n)$ to select the integration constant to be zero. Then $a_j, b_j, c_j (1 \leqslant j \leqslant n)$ can uniquely be determined by (2.3). A direct calculation gives

$$\begin{aligned}
&c_1 = q_x + r, \qquad a_1 = -\frac{q^2}{2}, \qquad b_1 = q_{xx} + r_x - \frac{q^3}{2} - qs, \\
&c_2 = q_{3x} - \frac{3}{2}q^2 q_x + r_{xx} - 2q_x s - qs_x - \frac{q^2 r}{2} - rs, \\
&a_2 = -\frac{(q^2)_{xx}}{2} + \frac{3}{2}q_x^2 + q_x r - qr_x + q^2 s + \frac{3}{8}q^4 + \frac{r^2}{2}, \\
&\ldots .
\end{aligned}$$

Substituting (a_j, b_j, c_j) into (2.4), we can get the nonlinear systems in the BPT hierarchy.

The first and second typical nonlinear systems ($n = 0, 1$) in the hierarchy are

$$q_{t_0} = q_x + r, \qquad r_{t_0} = -qs, \qquad s_{t_0} = -qr. \tag{2.5}$$

and

$$\begin{aligned} q_{t_1} &= q_{3x} - \frac{3}{2}q^2 q_x + r_{xx} - 2q_x s - q s_x - \frac{q^2 r}{2} - rs, \\ r_{t_1} &= -q_{xx} s - r_x s + \frac{q^3 s}{2} + q s^2, \\ s_{t_1} &= -q_{xx} r - r_x r + \frac{q^3 r}{2} + qrs. \end{aligned} \tag{2.6}$$

In the case when $r = s = 0$, the system (2.6) is reduced to MKdV equation

$$q_{t_1} = q_{3x} - \tfrac{3}{2}q^2 q_x.$$

3. Darboux transformation

In this section, we will construct a Darboux transformation for the soliton hierarchy (2.4). The Darboux transformation is actually a special gauge transformation

$$\widetilde{\psi} = T\psi \tag{3.1}$$

of the solutions of the Lax pairs (1.1) and (2.1). It is required that $\widetilde{\psi}$ also satisfies Lax pairs (1.1) and (2.1) with some $\widetilde{U}$ and $\widetilde{V}^{(n)}$, i.e.

$$\widetilde{\psi}_x = \widetilde{U}\widetilde{\psi}, \qquad \widetilde{U} = (T_x + TU)T^{-1}, \tag{3.2}$$

$$\widetilde{\psi}_t = \widetilde{V}^{(n)}\widetilde{\psi}, \qquad \widetilde{V}^{(n)} = (T_t + TV^{(n)})T^{-1}. \tag{3.3}$$

By cross differentiating (3.2) and (3.3), we get

$$\widetilde{U}_t - \widetilde{V}^{(n)}_x + [\widetilde{U}, \widetilde{V}^{(n)}] = T(U_t - V^{(n)}_x + [U, V^{(n)}])T^{-1}, \tag{3.4}$$

which imply that in order to make systems (2.4) invariant under the gauge transformation (3.1), we should require $\widetilde{U}, \widetilde{V}^{(n)}$ have the same forms as $U, V^{(n)}$ respectively. At the same time the old potentials q, r and s in $U, V^{(n)}$ will be mapped into new potentials $\widetilde{q}, \widetilde{r}$ and $\widetilde{s}$ in $\widetilde{U}, \widetilde{V}^{(n)}$. This process can be done continually and usually it may yield a series of multisoliton solutions. Following the idea of [2], we can construct the Darboux transformation for soliton hierarchy (2.4) as follows.

First, we define that

$$U_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad U_0 = \frac{1}{2}\begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix}, \qquad U_{-1} = \frac{1}{2}\begin{pmatrix} s & r \\ -r & -s \end{pmatrix}.$$

Then U can be written as

$$U = U_1\lambda + U_0 + U_{-1}\lambda^{-1}. \tag{3.5}$$

Let $h = (h_1, h_2)^T$ be a solution of spectral problem (1.1) and (2.1) when $\lambda = \lambda_0$ ($\lambda_0 \neq 0$). Then it is easy to see that $h^- = (h_2, h_1)^T$ is also a solution of the spectral problem (1.1) and (2.1) when $\lambda = -\lambda_0$. We construct a new matrix

$$H = (h, h^-).$$

By using (1.1) and (2.1), we can get that

$$H_x = U_1 H\Lambda + U_0 H + U_{-1} H\Lambda^{-1}, \qquad H_{t_n} = \sum_{j=0}^{n} A_j H\Lambda^{2(n-j)+1} + B_j H\Lambda^{2(n-j)}. \tag{3.6}$$

where

$$\Lambda = \begin{pmatrix} \lambda_0 & 0 \\ 0 & -\lambda_0 \end{pmatrix}.$$

We construct

$$T = \lambda I + S, \tag{3.7}$$

where

$$\begin{aligned} S &= -H\Lambda H^{-1} \\ &= \frac{1}{h_1^2 - h_2^2} \begin{pmatrix} -\lambda_0\left(h_1^2 + h_2^2\right) & 2\lambda_0 h_1 h_2 \\ -2\lambda_0 h_1 h_2 & \lambda_0\left(h_1^2 + h_2^2\right) \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix}. \end{aligned} \tag{3.8}$$

Substituting (3.7) into (3.2), we can get that

$$\widetilde{U} = \widetilde{U}_1\lambda + \widetilde{U}_0 + \widetilde{U}_{-1}\lambda^{-1}, \tag{3.9}$$

where $\widetilde{U}_1$, $\widetilde{U}_0$ and $\widetilde{U}_{-1}$ are determined by the following equations:

$$\begin{aligned} &\widetilde{U}_1 = U_1, && \widetilde{U}_0 = U_0 + SU_1 - \widetilde{U}_1 S, \\ &\widetilde{U}_{-1} = U_{-1} + S_x + SU_0 - \widetilde{U}_0 S, && \widetilde{U}_{-1} = SU_{-1}S^{-1}. \end{aligned} \tag{3.10}$$

With the help of (3.6) and the first two equations of (3.10), we can prove that the third equation of (3.10) is equivalent to the fourth one.

Substituting (3.7) into (3.3), and with the help of (3.6), we can get that

$$\widetilde{V}^{(n)} = \sum_{j=0}^{n} (\widetilde{A}_j \lambda^{2(n-j)+1} + \widetilde{B}_j \lambda^{2(n-j)}), \tag{3.11}$$

where $\widetilde{A}_j$ and $\widetilde{B}_j$ are determined by the following equations:

$$\begin{aligned} &\widetilde{A}_0 = A_0 \qquad \widetilde{B}_{j-1} = B_{j-1} + SA_{j-1} - \widetilde{A}_{j-1} S, \\ &\widetilde{A}_j = A_j + SB_{j-1} - \widetilde{B}_{j-1} S, \qquad 1 \leqslant j \leqslant n, \\ &\widetilde{B}_n = B_n + SA_n - \widetilde{A}_n S. \end{aligned} \tag{3.12}$$

Next we will prove that $\widetilde{U}$ and $\widetilde{V}^{(n)}$ have the same forms as U and $V^{(n)}$ after some transformations.

Proposition 1. *The matrix $\widetilde{U}$ determined by (3.9) has the same form as U, that is*

$$\begin{aligned} \widetilde{U} &= \widetilde{U}_1\lambda + \widetilde{U}_0 + \widetilde{U}_{-1}\lambda^{-1} \\ &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda + \frac{1}{2}\begin{pmatrix} 0 & \widetilde{q} \\ \widetilde{q} & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \widetilde{s} & \widetilde{r} \\ -\widetilde{r} & -\widetilde{s} \end{pmatrix}\lambda^{-1}, \end{aligned} \tag{3.13}$$

where the transformations between q, r, s and $\widetilde{q}, \widetilde{r}, \widetilde{s}$ are given by

$$\begin{aligned} \widetilde{q} &= q - 2\beta, \\ \widetilde{r} &= \frac{\alpha^2 + \beta^2}{-\alpha^2 + \beta^2} r - \frac{2\alpha\beta}{-\alpha^2 + \beta^2} s, \\ \widetilde{s} &= \frac{2\alpha\beta}{-\alpha^2 + \beta^2} r - \frac{\alpha^2 + \beta^2}{-\alpha^2 + \beta^2} s. \end{aligned} \tag{3.14}$$

α, β are determined by (3.8). The transformation $(\psi, q, r, s) \to (\widetilde{\psi}, \widetilde{q}, \widetilde{r}, \widetilde{s})$ is called a Darboux transformation of the spectral problem (1.1).

Proof. From (3.10), we obtain that

$$\widetilde{U}_1 = U_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\begin{aligned}
\widetilde{U}_0 &= U_0 + SU_1 - \widetilde{U}_1 S \\
&= \frac{1}{2}\begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 & q - 2\beta \\ q - 2\beta & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & \widetilde{q} \\ \widetilde{q} & 0 \end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
\widetilde{U}_{-1} &= SU_{-1}S^{-1} \\
&= \frac{1}{2}\begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix}\begin{pmatrix} s & r \\ -r & -s \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix}^{-1} \\
&= \frac{1}{2(-\alpha^2 + \beta^2)}\begin{pmatrix} -s(\alpha^2 + \beta^2) + 2r\alpha\beta & r(\alpha^2 + \beta^2) - 2s\alpha\beta \\ -r(\alpha^2 + \beta^2) + 2s\alpha\beta & s(\alpha^2 + \beta^2) - 2r\alpha\beta \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} \widetilde{s} & \widetilde{r} \\ -\widetilde{r} & -\widetilde{s} \end{pmatrix}.
\end{aligned}$$

So we get that after the transformation (3.14), $\widetilde{U}$ has the same form as U. The proof is completed. □

Next we will prove that $\widetilde{V}^{(n)}$ also has the same form as $V^{(n)}$ under the transformations (3.1) and (3.14).

Proposition 2. *The matrix $\widetilde{V}^{(n)}$ determined by (3.11) has the same form as $V^{(n)}$ under the transformations (3.1) and (3.14).*

Proof. Because $\widetilde{V}^{(n)}$ can be expressed as $\widetilde{V}^{(n)} = \sum_{j=0}^{n}(\widetilde{A}_j\lambda^{2(n-j)+1} + \widetilde{B}_j\lambda^{2(n-j)})$, we only need to prove that $\widetilde{A}_j$ and $\widetilde{B}_j$ have the same forms as A_j and B_j under the transformation (3.1) and (3.14).

From (3.12), we get that

$$\widetilde{A}_0 = A_0 = \frac{1}{2}\begin{pmatrix} a_0 & c_0 \\ -c_0 & -a_0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \widetilde{a}_0 & \widetilde{c}_0 \\ -\widetilde{c}_0 & -\widetilde{a}_0 \end{pmatrix}, \tag{3.15}$$

$$\begin{aligned}
\widetilde{B}_0 &= B_0 + SA_0 - \widetilde{A}_0 S \\
&= \frac{1}{2}\begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ -\beta & -\alpha \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 0 & q - 2\beta \\ q - 2\beta & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & \widetilde{b}_0 \\ \widetilde{b}_0 & 0 \end{pmatrix},
\end{aligned} \tag{3.16}$$

where

$$\widetilde{a}_0 = 1, \qquad \widetilde{c}_0 = 0, \qquad \widetilde{b}_0 = q - 2\beta = \widetilde{q}. \tag{3.17}$$

Thus we get that $\widetilde{A}_0$ and $\widetilde{B}_0$ have the same forms as A_0 and B_0 after the transformations.

Again by using (3.12), (3.15) and (3.16), through some calculations, we obtain that $\widetilde{A}_j$ and $\widetilde{B}_j (1 \leqslant j \leqslant n)$ have the following forms

$$\widetilde{A}_j = \frac{1}{2}\begin{pmatrix} \widetilde{a}_j & \widetilde{c}_j \\ -\widetilde{c}_j & -\widetilde{a}_j \end{pmatrix}, \qquad \widetilde{B}_j = \frac{1}{2}\begin{pmatrix} 0 & \widetilde{b}_j \\ \widetilde{b}_j & 0 \end{pmatrix}, \tag{3.18}$$

where

$$\begin{aligned} &\widetilde{a}_j = a_j + \beta(b_{j-1} + \widetilde{b}_{j-1}), \qquad \widetilde{c}_j = c_j + \alpha(b_{j-1} + \widetilde{b}_{j-1}), \\ &\widetilde{b}_j = b_j + \alpha(c_j + \widetilde{c}_j) - \beta(a_j + \widetilde{a}_j). \end{aligned} \tag{3.19}$$

Next we only need to prove that $\widetilde{a}_j, \widetilde{b}_j$ and $\widetilde{c}_j (1 \leqslant j \leqslant n)$ have the same forms as a_j, b_j and c_j after the transformation.

By using (2.2) and (3.4), we have

$$\widetilde{U}_t - \widetilde{V}_x^{(n)} + [\widetilde{U}, \widetilde{V}^{(n)}] = T\big(U_t - V_x^{(n)} + [U, V^{(n)}]\big)T^{-1} = 0, \tag{3.20}$$

which is equivalent to the following recurrence relations

$$\begin{pmatrix} 1 & 0 & 0 \\ \widetilde{q} & 0 & \partial \\ \partial & -1 & \widetilde{q} \end{pmatrix}\begin{pmatrix} \widetilde{c}_{j+1} \\ \widetilde{b}_{j+1} \\ \widetilde{a}_{j+1} \end{pmatrix} = \begin{pmatrix} -\widetilde{s} & \partial & \widetilde{r} \\ 0 & \widetilde{r} & 0 \\ 0 & \widetilde{s} & 0 \end{pmatrix}\begin{pmatrix} \widetilde{c}_j \\ \widetilde{b}_j \\ \widetilde{a}_j \end{pmatrix}, \qquad 0 \leqslant j \leqslant n-1 \tag{3.21}$$

and the equations

$$\begin{pmatrix} \widetilde{q}_{t_n} \\ \widetilde{r}_{t_n} \\ \widetilde{s}_{t_n} \end{pmatrix} = \begin{pmatrix} \widetilde{c}_{n+1} \\ -\widetilde{s}\,\widetilde{b}_n \\ -\widetilde{r}\,\widetilde{b}_n \end{pmatrix}. \tag{3.22}$$

From (3.21), we can easily prove that $\widetilde{c}_j|_{(\widetilde{q},\widetilde{r})=(0,0)} = \widetilde{b}_j|_{(\widetilde{q},\widetilde{r})=(0,0)} = 0 (1 \leqslant j \leqslant n)$. Again by using (3.21), (3.14) and (3.19), we have

$$\begin{aligned} \widetilde{a}_{jx}|_{(\widetilde{q},\widetilde{r})=(0,0)} &= -\widetilde{q}\,\widetilde{c}_j + \widetilde{r}\,\widetilde{b}_{j-1}|_{(\widetilde{q},\widetilde{r})=(0,0)} = 0 \\ &= \partial_x(a_j + \beta(b_{j-1} + \widetilde{b}_{j-1}))|_{(\widetilde{q},\widetilde{r})=(0,0)} \\ &= \partial_x(a_j + \beta b_{j-1})|_{(\widetilde{q},\widetilde{r})=(0,0)}, \qquad 1 \leqslant j \leqslant n. \end{aligned}$$

Therefore

$$\begin{aligned} \widetilde{a}_j|_{(\widetilde{q},\widetilde{r})=(0,0)} &= a_j + \beta b_{j-1}|_{(\widetilde{q},\widetilde{r})=(0,0)} \\ &= a_j + \frac{q}{2} b_{j-1}|_{(\widetilde{q},\widetilde{r})=(0,0)} = f(t). \end{aligned} \tag{3.23}$$

Note the fact $a_j|_{(q,r)=(0,0)} = b_j|_{(q,r)=(0,0)} = 0$, so the integral constant $f(t)$ must be zero, i.e.

$$\widetilde{a}_j|_{(\widetilde{q},\widetilde{r})=(0,0)} = 0, \qquad 1 \leqslant j \leqslant n.$$

We proved that $\widetilde{a}_j, \widetilde{b}_j, \widetilde{c}_j$ satisfies the same equations and the same boundary conditions with a_j, b_j, c_j, so they must have the same forms. The proof is completed. □

From propositions 1 and 2, we get the following theorem.

Theorem 1. *The solution (q, r, s) of the BPT hierarchy (2.4) are mapped into their new solution $(\widetilde{q}, \widetilde{r}, \widetilde{s})$ under the Darboux transformations (3.1) and (3.14), where α, β are given by (3.8).*

4. Applications of Darboux transformations

In this section, we will apply the Darboux transformation (3.14) to construct explicit solutions for the BPT hierarchy (2.4). As usual we make the Darboux transformation starting from a special solution of (2.4). We start from $q = q_0, r = r_0, s = s_0$, and we choose

$$h^{(k)} = \begin{pmatrix} h_1^{(k)} \\ h_2^{(k)} \end{pmatrix}, \qquad 1 \leqslant k \leqslant N, \tag{4.1}$$

as a solution of the Lax pairs (1.1) and (2.1) when $\lambda = \lambda_k$. Then we could construct the multisoliton solutions of (2.4) as follows.

First, we construct

$$\begin{aligned} H^{(1)} &= (h^{(1)}, h^{(1)-}), \qquad \Lambda^{(1)} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix}, \\ S^{(1)} &= -H^{(1)}\Lambda^{(1)}(H^{(1)})^{-1} \\ &= \frac{1}{h_1^{(1)2} - h_2^{(1)2}} \begin{pmatrix} -\lambda_1\left(h_1^{(1)2} + h_2^{(1)2}\right) & 2\lambda_1 h_1^{(1)} h_2^{(1)} \\ -2\lambda_1 h_1^{(1)} h_2^{(1)} & \lambda_1\left(h_1^{(1)2} + h_2^{(1)2}\right) \end{pmatrix} \\ &= \begin{pmatrix} \alpha^{(1)} & \beta^{(1)} \\ -\beta^{(1)} & -\alpha^{(1)} \end{pmatrix}. \end{aligned} \tag{4.2}$$

Then by use of theorem 1, we can get the new solution (q_1, r_1, s_1) of (2.4) from the following equations:

$$\begin{aligned} q_1 &= q_0 - 2\beta^{(1)}, \qquad r_1 = \frac{\alpha^{(1)2} + \beta^{(1)2}}{-\alpha^{(1)2} + \beta^{(1)2}} r_0 - \frac{2\alpha^{(1)}\beta^{(1)}}{-\alpha^{(1)2} + \beta^{(1)2}} s_0, \\ s_1 &= \frac{2\alpha^{(1)}\beta^{(1)}}{-\alpha^{(1)2} + \beta^{(1)2}} r_0 - \frac{\alpha^{(1)2} + \beta^{(1)2}}{-\alpha^{(1)2} + \beta^{(1)2}} s_0. \end{aligned} \tag{4.3}$$

With the help of (3.1), (3.7) and after some calculations, we can get the solution of Lax pairs (1.1) and (2.1), when $q = q_1, r = r_1, s = s_1$ and $\lambda = \lambda_2$. The solution can be expressed as

$$\overline{h}^{(2)} = \begin{pmatrix} \overline{h}_1^{(2)} \\ \overline{h}_2^{(2)} \end{pmatrix} = \frac{\lambda_1 + \lambda_2}{[h^{(1)}, h^{(1)}]} \begin{pmatrix} \begin{vmatrix} h_1^{(2)} & [h^{(2)}, h^{(1)}] \\ h_1^{(1)} & [h^{(1)}, h^{(1)}] \end{vmatrix} \\ \begin{vmatrix} h_2^{(2)} & [h^{(2)}, h^{(1)}] \\ h_2^{(1)} & [h^{(1)}, h^{(1)}] \end{vmatrix} \end{pmatrix}, \tag{4.4}$$

where

$$[h^{(i)}, h^{(j)}] = \frac{h_1^{(i)} h_1^{(j)} - h_2^{(i)} h_2^{(j)}}{\lambda_i + \lambda_j}.$$

We construct

$$\begin{aligned} H^{(2)} &= (\overline{h}^{(2)}, \overline{h}^{(2)-}), \qquad \Lambda^{(2)} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & -\lambda_2 \end{pmatrix}, \\ S^{(2)} &= -H^{(2)}\Lambda^{(2)}(H^{(2)})^{-1} \\ &= \frac{1}{\overline{h}_1^{(2)2} - \overline{h}_2^{(2)2}} \begin{pmatrix} -\lambda_2\left(\overline{h}_1^{(2)2} + \overline{h}_2^{(2)2}\right) & 2\lambda_2 \overline{h}_1^{(2)} \overline{h}_2^{(2)} \\ -2\lambda_2 \overline{h}_1^{(2)} \overline{h}_2^{(2)} & \lambda_2\left(\overline{h}_1^{(2)2} + \overline{h}_2^{(2)2}\right) \end{pmatrix} \\ &= \begin{pmatrix} \alpha^{(2)} & \beta^{(2)} \\ -\beta^{(2)} & -\alpha^{(2)} \end{pmatrix}. \end{aligned} \tag{4.5}$$

Then we can get the new solution (q_2, r_2, s_2) of (2.4) from the following equations:

$$q_2 = q_1 - 2\beta^{(2)}, \qquad r_2 = \frac{\alpha^{(2)^2} + \beta^{(2)^2}}{-\alpha^{(2)^2} + \beta^{(2)^2}} r_1 - \frac{2\alpha^{(2)}\beta^{(2)}}{-\alpha^{(2)^2} + \beta^{(2)^2}} s_1,$$
$$s_2 = \frac{2\alpha^{(2)}\beta^{(2)}}{-\alpha^{(2)^2} + \beta^{(2)^2}} r_1 - \frac{\alpha^{(2)^2} + \beta^{(2)^2}}{-\alpha^{(2)^2} + \beta^{(2)^2}} s_1. \tag{4.6}$$

If we have done the Darboux transformation $N - 1$ times and got the solution of (2.4) as $(q_{N-1}, r_{N-1}, s_{N-1})$, we can express the solution of Lax pairs (1.1) and (2.1) ($q = q_{N-1}$, $r = r_{N-1}, s = s_{N-1}, \lambda = \lambda_N$) as follows,

$$\overline{h}^{(N)} = \begin{pmatrix} \overline{h}_1^{(N)} \\ \overline{h}_2^{(N)} \end{pmatrix} = \Delta_N \begin{pmatrix} \begin{vmatrix} h_1^{(N)} & [h^{(N)}, h^{(1)}] & \cdots & [h^{(N)}, h^{(N-1)}] \\ h_1^{(1)} & [h^{(1)}, h^{(1)}] & \cdots & [h^{(1)}, h^{(N-1)}] \\ \vdots & \vdots & \ddots & \vdots \\ h_1^{(N-1)} & [h^{(N-1)}, h^{(1)}] & \cdots & [h^{(N-1)}, h^{(N-1)}] \end{vmatrix} \\ \begin{vmatrix} h_2^{(N)} & [h^{(N)}, h^{(1)}] & \cdots & [h^{(N)}, h^{(N-1)}] \\ h_2^{(1)} & [h^{(1)}, h^{(1)}] & \cdots & [h^{(1)}, h^{(N-1)}] \\ \vdots & \vdots & \ddots & \vdots \\ h_2^{(N-1)} & [h^{(N-1)}, h^{(1)}] & \cdots & [h^{(N-1)}, h^{(N-1)}] \end{vmatrix} \end{pmatrix}, \tag{4.7}$$

where

$$\Delta_N = \frac{(\lambda_N + \lambda_1)(\lambda_N + \lambda_2)\cdots(\lambda_N + \lambda_{N-1})}{\delta_{N-1}},$$
$$\delta_N = \begin{vmatrix} [h^{(1)}, h^{(1)}] & [h^{(1)}, h^{(2)}] & \cdots & [h^{(1)}, h^{(N)}] \\ [h^{(2)}, h^{(1)}] & [h^{(2)}, h^{(2)}] & \cdots & [h^{(2)}, h^{(N)}] \\ \vdots & \vdots & \ddots & \vdots \\ [h^{(N)}, h^{(1)}] & [h^{(N)}, h^{(2)}] & \cdots & [h^{(N)}, h^{(N)}] \end{vmatrix}. \tag{4.8}$$

We construct

$$H^{(N)} = (\overline{h}^{(N)}, \overline{h}^{(N)-}), \qquad \Lambda^{(N)} = \begin{pmatrix} \lambda_N & 0 \\ 0 & -\lambda_N \end{pmatrix},$$
$$S^{(N)} = -H^{(N)}\Lambda^{(N)}(H^{(N)})^{-1}$$
$$= \frac{1}{\overline{h}_1^{(N)2} - \overline{h}_2^{(N)2}} \begin{pmatrix} -\lambda_N\left(\overline{h}_1^{(N)2} + \overline{h}_2^{(N)2}\right) & 2\lambda_N \overline{h}_1^{(N)}\overline{h}_2^{(N)} \\ -2\lambda_N \overline{h}_1^{(N)}\overline{h}_2^{(N)} & \lambda_N\left(\overline{h}_1^{(N)2} + \overline{h}_2^{(N)2}\right) \end{pmatrix}$$
$$= \begin{pmatrix} \alpha^{(N)} & \beta^{(N)} \\ -\beta^{(N)} & -\alpha^{(N)} \end{pmatrix}. \tag{4.9}$$

Then we can get the new solution (q_N, r_N, s_N) of (2.4) from the following equations

$$q_N = q_{N-1} - 2\beta^{(N)},$$
$$r_N = \frac{\alpha^{(N)^2} + \beta^{(N)^2}}{-\alpha^{(N)^2} + \beta^{(N)^2}} r_{N-1} - \frac{2\alpha^{(N)}\beta^{(N)}}{-\alpha^{(N)^2} + \beta^{(N)^2}} s_{N-1},$$
$$s_N = \frac{2\alpha^{(N)}\beta^{(N)}}{-\alpha^{(N)^2} + \beta^{(N)^2}} r_{N-1} - \frac{\alpha^{(N)^2} + \beta^{(N)^2}}{-\alpha^{(N)^2} + \beta^{(N)^2}} s_{N-1}. \tag{4.10}$$

This process can be done continually and yield a series of soliton solutions of the BPT hierarchy in theory.

Next we will give a simple analysis about the requirements to be satisfied in order to get a regular N-soliton solution.

First, we need to suppose that q_0, r_0, s_0 and $h^{(k)}$ $(1 \leqslant k \leqslant N)$ are all the regular functions of (x, t). From (4.3) and note the fact $-\alpha^{(1)^2} + \beta^{(1)^2} = -\lambda_1^2$, it is easy to see that if $\alpha^{(1)}, \beta^{(1)}$ are the regular functions of (x, t), then (q_1, r_1, s_1) must be the regular solution for the BPT hierarchy (2.4). With the help of (4.2), we can get that

$$\alpha^{(1)} = \frac{-\lambda_1\left(h_1^{(1)^2} + h_2^{(1)^2}\right)}{h_1^{(1)^2} - h_2^{(1)^2}}, \qquad \beta^{(1)} = \frac{2\lambda_1 h_1^{(1)} h_2^{(1)}}{h_1^{(1)^2} - h_2^{(1)^2}}. \tag{4.11}$$

Thus if $h_1^{(1)^2} - h_2^{(1)^2}$ have no zero point, $\alpha^{(1)}, \beta^{(1)}$ must be the regular functions of (x, t). From the above analysis, we obtain that if we can choose suitable $h^{(1)}$ and λ_1 such that $\delta_1 = [h^{(1)}, h^{(1)}]$ have no zero point, the 1-soliton solution (q_1, r_1, s_1) must be the regular solution of (2.4).

Suppose we have got the regular 1-soliton solution (q_1, r_1, s_1) for (2.4). Then from (4.6) and note the fact $-\alpha^{(2)^2} + \beta^{(2)^2} = -\lambda_2^2$, we can see that if $\alpha^{(2)}, \beta^{(2)}$ are the regular functions of (x, t), the solution (q_2, r_2, s_2) must be the regular 2-soliton solution for (2.4). By using (4.4) and (4.5), we have

$$\begin{aligned}
\alpha^{(2)} &= \frac{-\lambda_2\left(\overline{h}_1^{(2)^2} + \overline{h}_2^{(2)^2}\right)}{\overline{h}_1^{(2)^2} - \overline{h}_2^{(2)^2}} \\
&= -\lambda_2 \frac{\begin{vmatrix} h_1^{(2)} & [h^{(2)}, h^{(1)}] \\ h_1^{(1)} & [h^{(1)}, h^{(1)}] \end{vmatrix}^2 + \begin{vmatrix} h_2^{(2)} & [h^{(2)}, h^{(1)}] \\ h_2^{(1)} & [h^{(1)}, h^{(1)}] \end{vmatrix}^2}{\begin{vmatrix} h_1^{(2)} & [h^{(2)}, h^{(1)}] \\ h_1^{(1)} & [h^{(1)}, h^{(1)}] \end{vmatrix}^2 - \begin{vmatrix} h_2^{(2)} & [h^{(2)}, h^{(1)}] \\ h_2^{(1)} & [h^{(1)}, h^{(1)}] \end{vmatrix}^2}, \\
\beta^{(2)} &= \frac{2\lambda_2 \overline{h}_1^{(2)} \overline{h}_2^{(2)}}{\overline{h}_1^{(2)^2} - \overline{h}_2^{(2)^2}} \\
&= 2\lambda_2 \frac{\begin{vmatrix} h_1^{(2)} & [h^{(2)}, h^{(1)}] \\ h_1^{(1)} & [h^{(1)}, h^{(1)}] \end{vmatrix} \times \begin{vmatrix} h_2^{(2)} & [h^{(2)}, h^{(1)}] \\ h_2^{(1)} & [h^{(1)}, h^{(1)}] \end{vmatrix}}{\begin{vmatrix} h_1^{(2)} & [h^{(2)}, h^{(1)}] \\ h_1^{(1)} & [h^{(1)}, h^{(1)}] \end{vmatrix}^2 - \begin{vmatrix} h_2^{(2)} & [h^{(2)}, h^{(1)}] \\ h_2^{(1)} & [h^{(1)}, h^{(1)}] \end{vmatrix}^2}.
\end{aligned} \tag{4.12}$$

Through some calculations we can get that

$$\begin{vmatrix} h_1^{(2)} & [h^{(2)}, h^{(1)}] \\ h_1^{(1)} & [h^{(1)}, h^{(1)}] \end{vmatrix}^2 - \begin{vmatrix} h_2^{(2)} & [h^{(2)}, h^{(1)}] \\ h_2^{(1)} & [h^{(1)}, h^{(1)}] \end{vmatrix}^2 = 2\lambda_2\delta_1\delta_2.$$

We can see if δ_1, δ_2 have no zero point, $\alpha^{(2)}, \beta^{(2)}$ must be the regular functions of (x, t). Thus we know that in order to get the regular 2-soliton solution (q_2, r_2, s_2), we only need to choose suitable $h^{(1)}, h^{(2)}$ and λ_1, λ_2 such that δ_1 and δ_2 have no zero point.

Do it continually and through the similar analysis as above, we can finally get that in order to get a regular N-soliton solution (q_N, r_N, s_N), we only need to choose suitable $h^{(1)}, \ldots, h^{(N)}$

and $\lambda_1, \ldots, \lambda_N$ such that $\delta_1, \ldots, \delta_N$ have no zero point, where $\delta_k (1 \leqslant k \leqslant N)$ are given in (4.8).

We should note that the requirements we obtained above may not be necessary and may not easily be used in practice.

In the end, we will give a simple example. We will construct the 1-soliton solution for the BPT hierarchy (2.4). Substituting $q = 0, r = 0, s = 1$ into the Lax pairs (1.1) and (2.1), we choose a basic solution corresponding to $\lambda = \lambda_1$ as follows

$$h^{(1)} = \begin{pmatrix} c_1\,\mathrm{e}^{\xi_1} \\ c_2\,\mathrm{e}^{-\xi_1} \end{pmatrix}, \tag{4.13}$$

where $\xi_1 = \frac{1}{2}\left(\lambda_1 + \frac{1}{\lambda_1}\right)x + \frac{1}{2}\lambda_1^{2n+1}t$ and c_1, c_2 are two constants. We construct

$$H^{(1)} = (h^{(1)}, h^{(1)-}) = \begin{pmatrix} c_1\,\mathrm{e}^{\xi_1} & c_2\,\mathrm{e}^{-\xi_1} \\ c_2\,\mathrm{e}^{-\xi_1} & c_1\,\mathrm{e}^{\xi_1} \end{pmatrix}, \qquad \Lambda^{(1)} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix},$$

$$\begin{aligned} S^{(1)} &= \begin{pmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & -\alpha_1 \end{pmatrix} = -H^{(1)}\Lambda^{(1)}(H^{(1)})^{-1} \\ &= \begin{pmatrix} -\lambda_1\left(c_1^2\,\mathrm{e}^{2\xi_1} + c_2^2\,\mathrm{e}^{-2\xi_1}\right) & 2\lambda_1 c_1 c_2 \\ -2\lambda_1 c_1 c_2 & \lambda_1\left(c_1^2\,\mathrm{e}^{2\xi_1} + c_2^2\,\mathrm{e}^{-2\xi_1}\right) \end{pmatrix} \frac{1}{c_1^2\,\mathrm{e}^{2\xi_1} - c_2^2\,\mathrm{e}^{-2\xi_1}}. \end{aligned}$$

Thus from (3.14), we can get

$$\begin{aligned} q_1 &= \frac{-4\lambda_1 c_1 c_2}{c_1^2\,\mathrm{e}^{2\xi_1} - c_2^2\,\mathrm{e}^{-2\xi_1}}, \qquad r_1 = -\frac{4c_1c_2\left(c_1^2\,\mathrm{e}^{2\xi_1} + c_2^2\,\mathrm{e}^{-2\xi_1}\right)}{\left(c_1^2\,\mathrm{e}^{2\xi_1} - c_2^2\,\mathrm{e}^{-2\xi_1}\right)^2}, \\ s_1 &= 1 + \frac{8c_1^2c_2^2}{\left(c_1^2\,\mathrm{e}^{2\xi_1} - c_2^2\,\mathrm{e}^{-2\xi_1}\right)^2}. \end{aligned} \tag{4.14}$$

If we choose different constants c_1, c_2, the properties of the solution (q_1, r_1, s_1) may change a lot. For example, if we choose $c_1 = 1, c_2 = 1$, we can get the solution as

$$q_1 = \frac{-4\lambda_1}{\mathrm{e}^{2\xi_1} - \mathrm{e}^{-2\xi_1}}, \qquad r_1 = -\frac{4(\mathrm{e}^{2\xi_1} + \mathrm{e}^{-2\xi_1})}{(\mathrm{e}^{2\xi_1} - \mathrm{e}^{-2\xi_1})^2}, \qquad s_1 = 1 + \frac{8}{(\mathrm{e}^{2\xi_1} - \mathrm{e}^{-2\xi_1})^2},$$

which is a singular solution. If we choose $c_1 = 1, c_2 = i$, we can get the solution as

$$q_1 = \frac{-4i\lambda_1}{\mathrm{e}^{2\xi_1} + \mathrm{e}^{-2\xi_1}}, \qquad r_1 = -\frac{4i(\mathrm{e}^{2\xi_1} - \mathrm{e}^{-2\xi_1})}{(\mathrm{e}^{2\xi_1} + \mathrm{e}^{-2\xi_1})^2}, \qquad s_1 = 1 + \frac{-8}{(\mathrm{e}^{2\xi_1} + \mathrm{e}^{-2\xi_1})^2},$$

which is a regular solution, but is not a real solution.

If we choose $n = 0$ and $n = 1$ in (4.14), we can get the 1-soliton solution for systems (2.5) and (2.6).

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